STATISTICS IN TRANSITION new series, December 2018
Vol. 19, No. 4, pp. 597–620, DOI 10.21307/stattrans-2018-032

ON A LESS CUMBERSOME METHOD OF ESTIMATION OF PARAMETERS OF LINDLEY DISTRIBUTION BY ORDER STATISTICS

M. R. Irshad[1], **R. Maya**[2]

ABSTRACT

In this article, we have derived suitable U-statistics from a sample of any size exceeding a specified integer to estimate the location and scale parameters of Lindley distribution without the evaluation of means, variances and co-variances of order statistics of an equivalent sample size arising from the corresponding standard form of distribution. The exact variances of the estimators have been also obtained.

Key words: Order statistics, Lindley distribution, Best linear unbiased estimator, U-statistics.

1. Introduction

Lindley (1958) suggested a distribution to illustrate the difference between fiducial distribution and posterior distribution, the following probability density function (pdf),

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\ x > 0,\ \theta > 0. \quad (1)$$

Ghitany et al. (2008) developed different properties of, Lindley distribution and showed that the Lindley distribution fits better than the exponential distribution based on the waiting times before service of the bank customers. Sankaran (1970) used, Lindley distribution as the mixing distribution of a Poisson parameter and the resulting distribution is known as the Poisson-Lindley distribution. Zakerzadeh and Dolati (2009) obtained a generalized Lindley distribution and discussed its various properties and applications. Ghitany et al. (2013) and Nadarajah et al. (2011) recently proposed extensions of the Lindley distribution named the generalized Lindley and power Lindley distributions respectively. A discrete form of Lindley distribution was introduced by Gómez and Ojeda (2011) by discretizing the continuous Lindley distribution. Ali et al. (2013) considered Bayesian analysis of the Lindley model via informative and non-informative priors under different loss functions. Elbatal and Elgarhy (2013) have investigated most of the statistical properties of the transmuted quasi-Lindley distribution. Kadilar and Cakmakyapan (2016) introduced in

[1]Department of Statistics, Cochin University of Science and Technology, Kochi-682 022, India.
E-mail: irshadm24@gmail.com
[2]Department of Statistics, University College, Trivandrum-695 034.
E-mail: mayasekharan@gmail.com

the literature, the Lindley family of distributions. Nedjar and Zehdoudi (2016) introduced gamma Lindley distribution and studied some important properties of their proposed generalization. Shibu and Irshad (2016) introduced extended version of generalized Lindley distribution, it includes all the existing Lindley models. Again, Irshad and Maya (2017) developed another form of generalization and elucidated various reliability properties of their proposed model.
Based on reparametrisation of (1.1), Sultan and Thubyani (2016) developed location scale extension of Lindley distribution and derived Best Linear Unbiased Estimators (BLUEs) of location and scale parameters based on order statistics, its pdf is given by

$$f(x) = \frac{\theta^2}{\sigma(1+\theta)}\left(1+\frac{x-\mu}{\sigma}\right)e^{-\theta\left(\frac{x-\mu}{\sigma}\right)}, \quad x>\mu \text{ and } \theta,\mu,\sigma>0. \tag{2}$$

Even though best linear unbiased estimation of location and scale parameters using order statistics (see, Lloyd (1952)) is a widely accepted method of estimation, one serious difficulty involved in the application of this method is that in order to obtain these estimators one requires the values of means, variances and covariances of the entire order statistics of a random sample of size n arising from the corresponding standard distribution. Thus, the results of Sultan and Thubyani (2016) cannot help one to obtain the BLUEs of μ and σ for larger values of n. However, if one obtains the BLUEs of μ and σ by order statistics based on small or moderate sample of size m and use this as kernel of degree m to construct appropriate U-statistics to estimate μ and σ, then these U-statistics are highly useful as they estimate the parameters explicitly. Moreover, these estimators are highly preferred as they utilize the optimality conditions of BLUE as well as U-statistics. Thomas and Sreekumar (2004) developed the concept of U-statistics by taking BLUE based on the order statistics of a random sample of size two as kernel of degree two to estimate the scale parameter of generalized exponential distribution. Again, Thomas and Sreekumar (2007) generalized the results of Thomas and Sreekumar (2004) to generate estimators based on U-statistics for the location and scale parameters of any distribution, by taking best linear functions of order statistics of a sample of size $m<n$ as kernels.
In the work of Sultan and Thubyani (2016), they did not mentioned the means, variances and the covariances of the order statistics arising from the standard form of (1.2). In the case of location scale family of distributions, a study based on order statistics, it is necessary to evaluate the moments of order statistics arising from the corresponding standard form of the distributions.
Hence, the main objective of this work is to evaluate the moments of order statistics arising from the standard form of (1.2) for some known values of the shape parameter θ. Using these values, we determine the best linear unbiased estimators based on small sample sizes of the location and scale parameters of (1.2) and use them to generate appropriate U-statistics for estimating those parameters for any sample sizes.

2. BLUEs of location and scale parameters of Lindley distribution using order statistics

Let $\mathbf{X} = (X_{1:m}, X_{2:m}, \cdots, X_{m:m})'$ be the vector of order statistics of a random sample of size m drawn from (1.2). Define $Y_{r:m}$=$\frac{X_{r:m}-\mu}{\sigma}, r = 1,2,\cdots,m$. Then, $Y_{r:m}, r = 1,2,\cdots,m$ are distributed as the order statistics of a random sample of size m drawn from the standard form of (1.2) with pdf $f_0(y)$. Let $\alpha = (\alpha_{1:m}, \alpha_{2:m}, \cdots, \alpha_{m:m})'$ and $V = ((v_{r,s:m}))$ be the vector of means and dispersion matrix of the vector of order statistics of a random sample of size m drawn from $f_0(y)$. Then, the BLUEs of μ and σ based on order statistics given by (see, Sultan and Thubyani (2016))

$$\hat{\mu} = -\frac{\alpha' V^{-1}(\mathbf{1}\ \alpha' - \alpha\ \mathbf{1}')V^{-1}}{(\alpha' V^{-1}\alpha)(\mathbf{1}' V^{-1}\mathbf{1}) - (\alpha' V^{-1}\mathbf{1})^2}\mathbf{X}, \tag{3}$$

$$\hat{\sigma} = \frac{\mathbf{1}' V^{-1}(\mathbf{1}\ \alpha' - \alpha\ \mathbf{1}')V^{-1}}{(\alpha' V^{-1}\alpha)(\mathbf{1}' V^{-1}\mathbf{1}) - (\alpha' V^{-1}\mathbf{1})^2}\mathbf{X}, \tag{4}$$

with variances given by

$$Var(\hat{\mu}) = \frac{(\alpha' V^{-1}\alpha)\sigma^2}{(\alpha' V^{-1}\alpha)(\mathbf{1}' V^{-1}\mathbf{1}) - (\alpha' V^{-1}\mathbf{1})^2}, \tag{5}$$

$$Var(\hat{\sigma}) = \frac{(\mathbf{1}' V^{-1}\mathbf{1})\sigma^2}{(\alpha' V^{-1}\alpha)(\mathbf{1}' V^{-1}\mathbf{1}) - (\alpha' V^{-1}\mathbf{1})^2}, \tag{6}$$

where **1** is a column vector of $1's$ of the same dimension as **X**.

2.2 **U-Statistics**

Let $X_1, X_2, \cdots, X_n$ be independent observations coming from a population with distribution function $F(x;\theta)$. Then, the U-statistic for the parameter θ with the symmetric kernel $h(\cdot)$ of degree m is defined as

$$U(X_1, X_2, \cdots, X_n) = \frac{1}{\binom{n}{m}} \sum_{\beta \in B} h(X_{\beta_1}, X_{\beta_2}, \cdots, X_{\beta_m}), \tag{7}$$

where $B = \{\beta/\beta = (\beta_1, \beta_2, \cdots, \beta_m), \beta_1 < \beta_2 < \cdots < \beta_m\}$ is one of the $\binom{n}{m}$ combinations of m integers chosen without replacement from the set $(1,2,\cdots,n)$. Suppose that

$$E[h(X_1, X_2, \cdots, X_m)] = \theta \ \text{ and } \ E[h^2(X_1, X_2, \cdots, X_m)] < \infty. \tag{8}$$

Let $h(X_1, X_2, \cdots, X_\omega, X_{\omega+1}, \cdots, X_m)$ and $h(X_1, X_2, \cdots, X_\omega, X_{m+1}, \cdots, X_{2m-\omega})$ be two random variables having exactly ω samples in common, $\omega = 1,2,\cdots,m$. Let $\xi_\omega^{(m)}$ be the covariance between these two random variables. Then, the variance of the U-statistic given in (2.5) as (see, Hoeffding (1948))

$$Var[U(X_1,X_2,\cdots,X_n)] = \frac{1}{\binom{n}{m}} \sum_{\omega=1}^{m} \binom{m}{\omega}\binom{n-m}{m-\omega} \xi_{\omega}^{(m)}. \tag{9}$$

3. Estimation of parameters of Lindley distribution using U-Statistics

Let the BLUE of μ as given in (2.1) be represented as

$$h_1(X_1,X_2,\cdots,X_m) = a_1X_{1:m} + a_2X_{2:m} + \cdots + a_mX_{m:m} \tag{10}$$

and that of σ given in (2.2) be represented as

$$h_2(X_1,X_2,\cdots,X_m) = d_1X_{1:m} + d_2X_{2:m} + \cdots + d_mX_{m:m}, \tag{11}$$

where $a_1,a_2,\cdots,a_m$ and $d_1,d_2,\cdots,d_m$ are constants. Now, we can easily write

$$U_{1:n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i}\binom{r-1}{i} a_{i+1}\right] X_{r:n} \tag{12}$$

as the U-statistic for estimating μ based on kernel given in (3.1) and

$$U_{2:n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i}\binom{r-1}{i} d_{i+1}\right] X_{r:n} \tag{13}$$

as the U-statistic for estimating μ based on kernel given in (3.2), where we define $\binom{r-1}{i} = 0$ for $i \geq r$ and $\binom{n-r}{m-1-i} = 0$ for $n-r < m-1-i$.
If we write
$\xi_{\omega}^{(m)}$=Cov$[h_1(X_1,X_2,\cdots,X_{\omega},X_{\omega+1},\cdots,X_m),h_1(X_1,X_2,\cdots,X_{\omega},X_{m+1},\cdots,X_{2m-\omega})]$, as the covariance between two $h_1(.)$ functions with exactly ω common observations and $\psi_{\omega}^{(m)}$ = Cov$[h_2(X_1,X_2,\cdots,X_{\omega},X_{\omega+1},\cdots,X_m),h_2(X_1,X_2,\cdots,X_{\omega},X_{m+1},\cdots,X_{2m-\omega})]$, as the covariance between two $h_2(.)$ functions with exactly ω common observations for $\omega = 1,2,\cdots,m$, then the variances of $U_{1:n}^{(m)}$ and $U_{2:n}^{(m)}$ are given by

$$Var[U_{1:n}^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{\omega=1}^{m} \binom{m}{\omega}\binom{n-m}{m-\omega} \xi_{\omega}^{(m)} \tag{14}$$

and

$$Var[U_{2:n}^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{\omega=1}^{m} \binom{m}{\omega}\binom{n-m}{m-\omega} \psi_{\omega}^{(m)}. \tag{15}$$

Clearly $\xi_m^{(m)} = V[h_1(X_1, X_2, \cdots, X_m)]$ and $\psi_m^{(m)} = V[h_2(X_1, X_2, \cdots, X_m)]$ and are given in (2.3) and (2.4) respectively. Now, we evaluate the values of $\xi_\omega^{(m)}$ and $\psi_\omega^{(m)}$ for $\omega = 1, 2, \cdots, m-1$, using the methodology developed by Thomas and Sreekumar (2008), as explained in the following steps.
Define the vectors b_{m+k} for $k = 1, 2, \cdots, m-1$ as

$$b'_{m+k} = \left[\frac{\sum_{i=0}^{m-1} \binom{m+k-1}{m-1-i}\binom{o}{i} a_{i+1}}{\binom{m+k}{m}}, \frac{\sum_{i=0}^{m-1} \binom{m+k-2}{m-1-i}\binom{1}{i} a_{i+1}}{\binom{m+k}{m}}, \cdots, \frac{\sum_{i=0}^{m-1} \binom{0}{m-1-i}\binom{m+k-1}{i} a_{i+1}}{\binom{m+k}{m}} \right] \quad (16)$$

and define $w_k = \binom{m+k}{m}(b'_{m+k}V_{m+k}b_{m+k})\sigma^2 - \xi_m^{(m)}, k = 1, 2, \cdots, m-1$ where V_{m+k} is the variance covariance matrix of the vector of order statistics of random sample of size $m+k$ drawn from the distribution with pdf $f_0(y)$ and $\xi_m^{(m)}$ defined as above. Define the matrix

$$H = \begin{bmatrix} 0 & 0 & \dots & 0 & \binom{m}{m-1}\binom{1}{1} \\ 0 & 0 & \dots & \binom{m}{m-2}\binom{2}{2} & \binom{m}{m-1}\binom{2}{1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \binom{m}{1}\binom{m-1}{m-1} & \binom{m}{2}\binom{m-1}{m-2} & \dots & \binom{m}{m-2}\binom{m-1}{2} & \binom{m}{m-1}\binom{m-1}{1} \end{bmatrix} \times \begin{bmatrix} \xi_1^{(m)} \\ \xi_2^{(m)} \\ \vdots \\ \xi_{m-1}^{(m)} \end{bmatrix} \quad (17)$$

and the vector $w = (w_1, w_2, \cdots, w_{m-1})'$. Then, the components $\xi_\omega^{(m)}, \omega = 1, 2, \cdots, m-1$, involved in (3.5) are solved from the following equations

$$\left(\xi_1^{(m)}, \xi_2^{(m)}, \cdots, \xi_{m-1}^{(m)}\right)' = H^{-1}W. \quad (18)$$

Similarly, the values of $\psi_\omega^{(m)}$, $\omega = 1, 2, \cdots, m-1$ can be obtained as

$$\left(\psi_1^{(m)}, \psi_2^{(m)}, \cdots, \psi_{m-1}^{(m)}\right)' = H^{-1}Z, \quad (19)$$

where $Z' = (z_1, z_2, \cdots, z_{m-1})'$ with $z_k = \binom{m+k}{m}(g'_{m+k}V_{m+k}g_{m+k})\sigma^2 - \psi_m^{(m)}$ and g_{m+k} is obtained from (3.7) just by replacing each a_i by d_i, $i = 1, 2, \cdots, m$.

Once we obtain the values of $\xi_\omega^{(m)}, \psi_\omega^{(m)}, \omega = 1, 2, \cdots, m-1$, from (3.9) and (3.10) respectively, then the exact variances of the U-statistics for estimating μ and σ

based on any sample of size n can be obtained by using (3.5) and (3.6) without any further direct evaluation of moments of order statistics.

Table 1: Means of order statistics arising from the standard form of (1.2) for $n = 2(1)10$ and for $\theta = 0.50(0.50)2$.

n	r	$\theta = 0.50$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
2	1	1.88889	0.81250	0.49333	0.34722
	2	4.77778	2.18750	1.37333	0.98611
3	1	1.35254	0.56481	0.33798	0.23594
	2	2.96159	1.30787	0.80405	0.56979
	3	5.68587	2.62731	1.65798	1.19427
4	1	1.06481	0.43506	0.25773	0.17896
	2	2.21571	0.95408	0.57870	0.40687
	3	3.70748	1.66166	1.02940	0.73271
	4	6.34534	2.94920	1.86750	1.34813
5	1	0.88297	0.35464	0.20854	0.14425
	2	1.79219	0.75673	0.45452	0.31781
	3	2.85098	1.25011	0.76498	0.54045
	4	4.27847	1.93602	1.20568	0.86088
	5	6.86205	3.20250	2.03296	1.46994
6	1	0.75669	0.29971	0.17522	0.12086
	2	1.51439	0.62928	0.37511	0.26121
	3	2.34780	1.01164	0.61334	0.43101
	4	3.35416	1.48858	0.91661	0.64990
	5	4.74063	2.15975	1.35021	0.96637
	6	7.28634	3.41105	2.16951	1.57065
7	1	0.66343	0.25973	0.15114	0.10402
	2	1.31625	0.53963	0.31971	0.22192
	3	2.00972	0.85341	0.51361	0.35943
	4	2.79857	1.22262	0.74633	0.52644
	5	3.77085	1.68804	1.04432	0.74250
	6	5.12854	2.34843	1.47257	1.05592
	7	7.64597	3.58815	2.28566	1.65644
8	1	0.59150	0.22927	0.13291	0.09131
	2	1.16693	0.47290	0.27876	0.19300
	3	1.76422	0.73981	0.44254	0.30868
	4	2.41889	1.04273	0.63204	0.44402
	5	3.17826	1.40251	0.86061	0.60885
	6	4.12640	1.85937	1.15454	0.82268
	7	5.46259	2.51145	1.57858	1.13367
	8	7.95788	3.74196	2.38667	1.73112
9	1	0.53420	0.20528	0.11862	0.08137
	2	1.04989	0.42118	0.24722	0.17080
	3	1.57658	0.65390	0.38915	0.27070
	4	2.13951	0.91164	0.54933	0.38464

Table 1: Continued

n	r	$\theta=0.50$	$\theta=1$	$\theta=1.5$	$\theta=2$
9	5	2.76810	1.20660	0.73544	0.51825
	6	3.50639	1.55924	0.96075	0.68134
	7	4.43640	2.00943	1.25144	0.89335
	8	5.75579	2.65488	1.67205	1.20233
	9	8.23314	3.87785	2.47600	1.79722
10	1	0.48740	0.18589	0.10712	0.07339
	2	0.95540	0.37987	0.22215	0.15321
	3	1.42781	0.58645	0.34748	0.24117
	4	1.92371	0.81128	0.48639	0.33963
	5	2.46322	1.06217	0.64373	0.45215
	6	3.07298	1.35102	0.82715	0.58435
	7	3.79534	1.69805	1.04982	0.74600
	8	4.71115	2.14288	1.33784	0.95650
	9	6.01695	2.78288	1.75561	1.26379
	10	8.47938	3.99951	2.55604	1.85650

Table 2: Variances and covariances $v_{r,s:n}$ of order statistics arising from the standard form of (1.2) for $1 \leq r \leq s \leq n$, $n = 2(1)10$ and $\theta = 0.50(0.50)2$.

n	r	s	$\theta=0.50$	$\theta=1$	$\theta=1.5$	$\theta=2$
2	1	1	2.43210	0.52734	0.20996	0.10860
	1	2	2.08642	0.47266	0.19360	0.10204
	2	2	8.50617	2.02734	0.86062	0.46508
3	1	1	1.26895	0.26123	0.10077	0.05110
	1	2	1.10362	0.23853	0.09463	0.04881
	1	3	0.97997	0.21783	0.08823	0.04618
	2	2	3.03236	0.69148	0.28350	0.14931
	2	3	2.71450	0.63480	0.26528	0.14165
	3	3	8.76918	2.11496	0.90612	0.49298
4	1	1	0.80135	0.15814	0.05957	0.02978
	1	2	0.70230	0.14593	0.05649	0.02869
	1	3	0.63270	0.13553	0.05354	0.02756
	1	4	0.57352	0.12535	0.05032	0.02622
	2	2	1.67832	0.36848	0.14711	0.07609
	2	3	1.51875	0.34312	0.13966	0.07316
	2	4	1.38116	0.31802	0.13147	0.06969
	3	3	3.27371	0.76416	0.31833	0.16944
	3	4	2.99528	0.71127	0.30057	0.16174
	4	4	8.86142	2.15079	0.92644	0.50614
5	1	1	0.56080	0.10681	0.03949	0.01954
	1	2	0.49439	0.09931	0.03770	0.01892
	1	3	0.44832	0.09303	0.03603	0.01831
	1	4	0.41216	0.08737	0.03437	0.01765
	1	5	0.37866	0.08152	0.03250	0.01687

Table 2: Continued

n	r	s	$\theta = 0.50$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
5	2	2	1.10224	0.23411	0.09149	0.04668
	2	3	1.00239	0.21965	0.08751	0.04519
	2	4	0.92339	0.20655	0.08356	0.04361
	2	5	0.84967	0.19293	0.07907	0.04169
	3	3	1.86983	0.42397	0.17271	0.09046
	3	4	1.72821	0.39952	0.16512	0.08738
	3	5	1.59442	0.37388	0.15646	0.08361
	4	4	3.39454	0.80276	0.33772	0.18102
	4	5	3.14698	0.75391	0.32084	0.17354
	5	5	8.89317	2.16701	0.93675	0.51323
6	1	1	0.41851	0.07732	0.02816	0.01381
	1	2	0.37082	0.07232	0.02701	0.01343
	1	3	0.33770	0.06814	0.02595	0.01306
	1	4	0.31233	0.06447	0.02494	0.01267
	1	5	0.29088	0.06102	0.02391	0.01226
	1	6	0.26976	0.05731	0.02271	0.01175
	2	2	0.79383	0.16373	0.06287	0.03173
	2	3	0.72437	0.15443	0.06044	0.03085
	2	4	0.67089	0.14623	0.05810	0.02996
	2	5	0.62547	0.13850	0.05572	0.02900
	2	6	0.58056	0.13016	0.05295	0.02780
	3	3	1.25600	0.27740	0.11901	0.05736
	3	4	1.16581	0.26301	0.10670	0.05573
	3	5	1.08866	0.24937	0.10241	0.05396
	3	6	1.01185	0.23457	0.09738	0.05176
	4	4	1.97728	0.45681	0.18852	0.09961
	4	5	1.85124	0.43388	0.18114	0.09652
	4	6	1.72441	0.40879	0.17245	0.09267
	5	5	3.46241	0.82558	0.34965	0.18834
	5	6	3.23830	0.78023	0.33365	0.18114
	6	6	8.89922	2.17434	0.94229	0.51735
7	1	1	0.32640	0.05873	0.02112	0.01029
	1	2	0.29052	0.05520	0.02034	0.01004
	1	3	0.26544	0.05224	0.01962	0.00979
	1	4	0.24638	0.04966	0.01894	0.00954
	1	5	0.23077	0.04732	0.01827	0.00928
	1	6	0.21685	0.04504	0.01758	0.00900
	1	7	0.20251	0.04251	0.01676	0.00865
	2	2	0.60587	0.12173	0.04604	0.02304
	2	3	0.55438	0.11528	0.04443	0.02247
	2	4	0.51511	0.10966	0.04291	0.02191
	2	5	0.48283	0.10453	0.04141	0.02132
	2	6	0.45398	0.09954	0.03985	0.02068
	2	7	0.42417	0.09399	0.03800	0.01988

Table 2: Continued

n	r	s	$\theta=0.50$	$\theta=1$	$\theta=1.5$	$\theta=2$
7	3	3	0.92024	0.19842	0.07807	0.03996
	3	4	0.85639	0.18890	0.07543	0.03897
	3	5	0.80366	0.18020	0.07283	0.03794
	3	6	0.75633	0.17170	0.07012	0.03682
	3	7	0.70722	0.16222	0.06688	0.03540
	4	4	1.34807	0.30481	0.12374	0.06462
	4	5	1.26723	0.29107	0.11956	0.06295
	4	6	1.19420	0.27760	0.11518	0.06111
	4	7	1.11796	0.26249	0.10994	0.05878
	5	5	2.04404	0.47796	0.19905	0.10584
	5	6	1.93035	0.45653	0.19196	0.10282
	5	7	1.81053	0.43230	0.18342	0.09897
	6	6	3.50309	0.84002	0.35749	0.19328
	6	7	3.29717	0.79760	0.34229	0.18635
	7	7	8.89323	2.17717	0.94531	0.51984
8	1	1	0.26288	0.04621	0.01644	0.00797
	1	2	0.23494	0.04361	0.01588	0.00779
	1	3	0.21526	0.04142	0.01537	0.00761
	1	4	0.20032	0.03951	0.01489	0.00744
	1	5	0.18824	0.03781	0.01442	0.00727
	1	6	0.17786	0.03621	0.01396	0.00709
	1	7	0.16822	0.03462	0.01347	0.00689
	1	8	0.15794	0.03280	0.01287	0.00663
	2	2	0.48128	0.09443	0.03526	0.01751
	2	3	0.44144	0.08973	0.03413	0.01713
	2	4	0.41113	0.08565	0.03307	0.01674
	2	5	0.38656	0.08197	0.03204	0.01636
	2	6	0.36540	0.07853	0.03102	0.01595
	2	7	0.34574	0.07510	0.02993	0.01550
	2	8	0.32471	0.07118	0.02862	0.01493
	3	3	0.71207	0.15018	0.05827	0.02957
	3	4	0.66396	0.14343	0.05647	0.02892
	3	5	0.62483	0.13734	0.05474	0.02825
	3	6	0.59102	0.13163	0.05300	0.02756
	3	7	0.55952	0.12593	0.05116	0.02679
	3	8	0.52575	0.11940	0.04893	0.02580
	4	4	0.99933	0.22147	0.08862	0.04584
	4	5	0.94159	0.21223	0.08594	0.04480
	4	6	0.89149	0.20353	0.08325	0.04371
	4	7	0.84464	0.19481	0.08039	0.04250
	4	8	0.79421	0.18481	0.07691	0.04095

Table 2: Continued

n	r	s	$\theta = 0.50$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
8	5	5	1.40849	0.32343	0.13274	0.06983
	5	6	1.33539	0.31044	0.12866	0.06816
	5	7	1.26666	0.29739	0.12431	0.06630
	5	8	1.19225	0.28234	0.11900	0.06391
	6	6	2.08826	0.49242	0.20644	0.11030
	6	7	1.98435	0.47232	0.19966	0.10736
	6	8	1.87089	0.44900	0.19131	0.10357
	7	7	3.52834	0.84958	0.36289	0.19676
	7	8	3.33691	0.80962	0.34839	0.19009
	8	8	8.88133	2.17755	0.94689	0.52138
9	1	1	0.21698	0.03735	0.01317	0.00635
	1	2	0.19465	0.03538	0.01276	0.00622
	1	3	0.17878	0.03370	0.01237	0.00609
	1	4	0.16672	0.03225	0.01202	0.00597
	1	5	0.15702	0.03095	0.01168	0.00585
	1	6	0.14881	0.02975	0.01134	0.00572
	1	7	0.14150	0.02860	0.01101	0.00559
	1	8	0.13451	0.02744	0.01064	0.00544
	1	9	0.12683	0.02609	0.01020	0.00525
	2	2	0.39367	0.07560	0.02791	0.01378
	2	3	0.36188	0.07204	0.02709	0.01350
	2	4	0.33766	0.06895	0.02631	0.01323
	2	5	0.31816	0.06619	0.02556	0.01295
	2	6	0.30164	0.06364	0.02484	0.01267
	2	7	0.28690	0.06121	0.02410	0.01238
	2	8	0.27278	0.05873	0.02331	0.01205
	2	9	0.25726	0.05584	0.02234	0.01163
	3	3	0.57214	0.11823	0.04531	0.02283
	3	4	0.53434	0.11321	0.04402	0.02237
	3	5	0.50382	0.10871	0.04279	0.02191
	3	6	0.47792	0.10456	0.04158	0.02144
	3	7	0.45475	0.10059	0.04036	0.02095
	3	8	0.43253	0.09654	0.03904	0.02039
	3	9	0.40804	0.09181	0.03742	0.01967
	4	4	0.78068	0.16980	0.06708	0.03441
	4	5	0.73678	0.16314	0.06522	0.03371
	4	6	0.69940	0.15698	0.06340	0.03299
	4	7	0.66588	0.15107	0.06155	0.03224
	4	8	0.63365	0.14504	0.05956	0.03139
	4	9	0.59805	0.13798	0.05710	0.03029

Table 2: Continued

n	r	s	$\theta = 0.50$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
9	5	5	1.05313	0.23773	0.09631	0.05020
	5	6	1.00070	0.22890	0.09365	0.04915
	5	7	0.95351	0.22040	0.09095	0.04804
	5	8	0.90796	0.21170	0.08804	0.04679
	5	9	0.85748	0.20149	0.08443	0.04516
	6	6	1.45052	0.33672	0.13932	0.07370
	6	7	1.38371	0.32446	0.13537	0.07206
	6	8	1.31893	0.31188	0.13111	0.07021
	6	9	1.24675	0.29706	0.12581	0.06780
	7	7	2.11883	0.50271	0.21184	0.11361
	7	8	2.02279	0.48377	0.20534	0.11076
	7	9	1.91493	0.46133	0.19722	0.10703
	8	8	3.54422	0.85611	0.36673	0.19930
	8	9	3.36461	0.81822	0.35285	0.19286
	9	9	8.86655	2.17654	0.94760	0.52231
10	1	1	0.18260	0.03085	0.01079	0.00518
	1	2	0.16438	0.02931	0.01047	0.00508
	1	3	0.15131	0.02799	0.01018	0.00499
	1	4	0.14135	0.02685	0.00991	0.00490
	1	5	0.13335	0.02583	0.00965	0.00481
	1	6	0.12664	0.02490	0.00940	0.00471
	1	7	0.12078	0.02402	0.00915	0.00462
	1	8	0.11540	0.02317	0.00890	0.00452
	1	9	0.11013	0.02229	0.00862	0.00440
	1	10	0.10420	0.02125	0.00828	0.00425
	2	2	0.32931	0.06200	0.02267	0.01113
	2	3	0.30334	0.05924	0.02204	0.01092
	2	4	0.28349	0.05683	0.02146	0.01072
	2	5	0.26755	0.05468	0.02090	0.01052
	2	6	0.25417	0.05272	0.02035	0.01032
	2	7	0.24245	0.05087	0.01982	0.01011
	2	8	0.23169	0.04908	0.01927	0.00989
	2	9	0.22115	0.04722	0.01868	0.00964
	2	10	0.20928	0.04501	0.01793	0.00931
	3	3	0.47258	0.09582	0.03633	0.01818
	3	4	0.44199	0.09195	0.03536	0.01785
	3	5	0.41736	0.08850	0.03445	0.01751
	3	6	0.39665	0.08534	0.03356	0.01718
	3	7	0.37849	0.08237	0.03268	0.01683
	3	8	0.36180	0.07948	0.03178	0.01647
	3	9	0.34541	0.07648	0.03080	0.01605
	3	10	0.32694	0.07292	0.02958	0.01551

Table 2: Continued

n	r	s	$\theta = 0.50$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
10	4	4	0.63230	0.13513	0.05278	0.02687
	4	5	0.59751	0.13010	0.05142	0.02638
	4	6	0.56818	0.12550	0.05010	0.02588
	4	7	0.54242	0.12117	0.04880	0.02536
	4	8	0.51870	0.11694	0.04747	0.02481
	4	9	0.49536	0.11256	0.04601	0.02419
	4	10	0.46901	0.10734	0.04419	0.02337
	5	5	0.82859	0.18404	0.07369	0.03811
	5	6	0.78852	0.17761	0.07182	0.03739
	5	7	0.75322	0.17155	0.06997	0.03665
	5	8	0.72064	0.16562	0.06807	0.03586
	5	9	0.68851	0.15946	0.06600	0.03497
	5	10	0.65216	0.15212	0.06341	0.03380
	6	6	1.09177	0.24970	0.10210	0.05356
	6	7	1.04376	0.24130	0.09950	0.05251
	6	8	0.99930	0.23307	0.09683	0.05139
	6	9	0.95532	0.22450	0.09392	0.05012
	6	10	0.90539	0.21426	0.09026	0.04846
	7	7	1.48097	0.34656	0.14430	0.07668
	7	8	1.41930	0.33497	0.14049	0.07507
	7	9	1.35803	0.32286	0.13633	0.07324
	7	10	1.28812	0.30833	0.13108	0.07084
	8	8	2.14058	0.51027	0.21590	0.11614
	8	9	2.05101	0.49234	0.20966	0.11338
	8	10	1.94806	0.47070	0.20176	0.10973
	9	9	3.55411	0.86065	0.36954	0.20121
	9	10	3.38433	0.82453	0.35619	0.19498
	10	10	8.85046	2.17473	0.94776	0.52286

Table 3: Coefficients of $X_{i:n}$ in the BLUE $\hat{\mu}$ and $V_1 = \frac{Var(\hat{\mu})}{\sigma^2}$.

n	θ	Coefficients										V_1
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
2		1.65385	-0.65385									5.77647
3		1.41191	-0.15873	-0.25319								2.19107
4		1.28908	-0.04630	-0.10253	-0.14024							1.20397
5		1.21488	-0.00631	-0.04601	-0.07171	-0.09085						0.77759
6	0.50	1.16548	0.01068	-0.01937	-0.03931	-0.05313	-0.06435					0.55037
7		1.13047	0.01841	-0.00528	-0.02145	-0.03276	-0.04107	-0.04831				0.41330
8		1.10449	0.02197	0.00266	-0.01081	-0.02035	-0.02738	-0.03279	-0.03779			0.32350
9		1.08462	0.02336	0.00736	-0.00412	-0.01237	-0.01842	-0.02310	-0.02685	-0.03047		0.26112
10		1.06900	0.02364	0.01017	0.00021	-0.00696	-0.01231	-0.01646	-0.01971	-0.02243	-0.02515	0.21582
2		1.59091	-0.59091									1.15391
3		1.37351	-0.15579	-0.21772								0.42056
4		1.26736	-0.06025	-0.09082	-0.1163							0.22354
5		1.20501	-0.02638	-0.04537	-0.06009	-0.07317						0.14033
6	1	1.16434	-0.01162	-0.02455	-0.03470	-0.04289	-0.05058					0.09690
7		1.13592	-0.00437	-0.01376	-0.02114	-0.02725	-0.03222	-0.03718				0.07121
8		1.11503	-0.00053	-0.00772	-0.01317	-0.01808	-0.02183	-0.02516	-0.02855			0.05467
9		1.09930	0.00128	-0.00375	-0.00861	-0.01224	-0.01528	-0.01785	-0.02020	-0.02265		0.04336
10		1.08676	0.00250	-0.00145	-0.00548	-0.00839	-0.01101	-0.01300	-0.01492	-0.01659	-0.01842	0.03527

Table 3: Continued

n	θ	Coefficients										V_1
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
2		1.56060	-0.56060									0.44307
3		1.35722	-0.15639	-0.20083								0.15750
4		1.25905	-0.06732	-0.08654	-0.10520							0.08212
5		1.20174	-0.03502	-0.04602	-0.05547	-0.06523						0.05077
6	1.5	1.16417	-0.02017	-0.02718	-0.03343	-0.03881	-0.04457					0.03462
7		1.13813	-0.01263	-0.01739	-0.02167	-0.02526	-0.02873	-0.03246				0.02517
8		1.11867	-0.00787	-0.01177	-0.01495	-0.01730	-0.01992	-0.02215	-0.02471			0.01915
9		1.10428	-0.00623	-0.00731	-0.01074	-0.01269	-0.01410	-0.01616	-0.01755	-0.01949		0.01507
10		1.09199	-0.00323	-0.00577	-0.00783	-0.00918	-0.01075	-0.01188	-0.01326	-0.01432	-0.01576	0.01218
2		1.54347	-0.54347									0.22490
3		1.34885	-0.15753	-0.19132								0.07868
4		1.25518	-0.07150	-0.08463	-0.09904							0.04055
5		1.20010	-0.03918	-0.04681	-0.05315	-0.06096						0.02486
6	2	1.16451	-0.02438	-0.02931	-0.03266	-0.03680	-0.04135					0.01682
7		1.13961	-0.01692	-0.01933	-0.02202	-0.02441	-0.02696	-0.02998				0.01217
8		1.12049	-0.01204	-0.01306	-0.01553	-0.01744	-0.01901	-0.02069	-0.02273			0.00922
9		1.10628	-0.00907	-0.00930	-0.01160	-0.01316	-0.01374	-0.01523	-0.01631	-0.01785		0.00722
10		1.09452	-0.00546	-0.00811	-0.00887	-0.01041	-0.00992	-0.01172	-0.01246	-0.01318	-0.01438	0.00581

Table 4: Coefficients of $X_{i:n}$ in the BLUE $\hat{\sigma}$ and $V_2 = \frac{Var(\hat{\sigma})}{\sigma^2}$.

n	θ	Coefficients										V_2
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
2		-0.34615	0.34615									0.81065
3		-0.31956	0.14123	0.17833								0.39436
4		-0.30376	0.07960	0.10436	0.11980							0.25806
5		-0.29298	0.05104	0.07024	0.08177	0.08993						0.19089
6	0.50	-0.28508	0.03508	0.05081	0.06049	0.06686	0.07184					0.15107
7		-0.27900	0.02514	0.03846	0.04686	0.05243	0.05638	0.05973				0.12479
8		-0.27416	0.01847	0.03002	0.03747	0.04247	0.04601	0.04867	0.05107			0.10619
9		-0.27022	0.01380	0.02394	0.03064	0.03521	0.03844	0.04086	0.04276	0.04457		0.09233
10		-0.26692	0.01037	0.01941	0.02550	0.02969	0.03270	0.03495	0.03668	0.03811	0.03951	0.08163
2		-0.72727	0.72727									0.851234
3		-0.68078	0.30627	0.37451								0.41765
4		-0.65430	0.18069	0.22097	0.25264							0.27504
5		-0.63673	0.12249	0.15124	0.17248	0.19051						0.20448
6	1	-0.62410	0.08983	0.11187	0.12838	0.14121	0.15282					0.16250
7		-0.61453	0.06931	0.08696	0.10041	0.11093	0.11941	0.12752				0.13471
8		-0.60700	0.05542	0.07004	0.08118	0.09026	0.09735	0.10337	0.10938			0.11498
9		-0.60097	0.04566	0.05765	0.06758	0.07522	0.08145	0.08659	0.09108	0.09574		0.10025
10		-0.59584	0.03818	0.04862	0.05718	0.06392	0.06952	0.07400	0.07794	0.08138	0.08511	0.08885

Table 4: Continued

n	θ	Coefficients										V_2
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
2		-1.13636	1.13636									0.88246
3		-1.07630	0.49268	0.58362								0.43554
4		-1.04318	0.30025	0.34879	0.39414							0.28802
5		-1.02177	0.21044	0.24305	0.27052	0.29777						0.21479
6	1.5	-1.00639	0.15937	0.18309	0.20354	0.22106	0.23932					0.17112
7		-0.99510	0.12706	0.14532	0.16110	0.17464	0.18691	0.20007				0.14215
8		-0.98601	0.10438	0.11963	0.13226	0.14304	0.15294	0.16188	0.17187			0.12153
9		-0.97930	0.08942	0.09999	0.11152	0.12061	0.12839	0.13600	0.14270	0.15066		0.10612
10		-0.97258	0.07576	0.08675	0.09576	0.10330	0.11052	0.11646	0.12227	0.12765	0.13410	0.09416
2		-1.56521	1.56521									0.90548
3		-1.49633	0.69494	0.80139								0.44863
4		-1.45966	0.43336	0.48533	0.54096							0.29740
5		-1.43583	0.30938	0.34338	0.37411	0.40896						0.22223
6	2	-1.41966	0.23860	0.26324	0.28398	0.30498	0.32885					0.17729
7		-1.40784	0.19391	0.21121	0.22751	0.24264	0.25744	0.27513				0.14744
8		-1.39773	0.16223	0.17507	0.18858	0.20064	0.21177	0.22295	0.23650			0.12618
9		-1.39056	0.13970	0.14879	0.16078	0.17059	0.17896	0.18782	0.19649	0.20743		0.11024
10		-1.38371	0.11957	0.13164	0.13904	0.14854	0.15352	0.16222	0.16881	0.17566	0.18471	0.09789

Table 5: Values of $\xi_{\omega}^{(m)}$, for $\omega = 1,2,\cdots,m$ and $m = 2(1)5$.

		$\theta = 0.50$	$\theta = 1$	$\theta = 1.50$	$\theta = 2$
m	ω	$\sigma^{-2}\xi_{\omega}^{(m)}$	$\sigma^{-2}\xi_{\omega}^{(m)}$	$\sigma^{-2}\xi_{\omega}^{(m)}$	$\sigma^{-2}\xi_{\omega}^{(m)}$
2	1	1.05101	0.20177	0.07556	0.03792
	2	5.77647	0.15391	0.44307	0.22490
3	1	0.33650	0.06094	0.02213	0.01089
	2	0.99730	0.18551	0.06822	0.03370
	3	2.19107	0.42056	0.15750	0.07868
4	1	0.15227	0.02644	0.00933	0.00460
	2	0.37925	0.06701	0.02399	0.01165
	3	0.71139	0.12854	0.04652	0.02278
	4	1.20397	0.22354	0.08212	0.04055
5	1	0.08272	0.01377	0.00493	0.00217
	2	0.19121	0.03245	0.01138	0.00548
	3	0.33382	0.05766	0.02039	0.00987
	4	0.52247	0.09206	0.03290	0.01598
	5	0.77759	0.14033	0.05077	0.02486

Table 6: Values of $\psi_{\omega}^{(m)}$, for $\omega = 1,2,\cdots,m$ and $m = 2(1)5$.

		$\theta = 0.50$	$\theta = 1$	$\theta = 1.50$	$\theta = 2$
m	ω	$\sigma^{-2}\psi_{\omega}^{(m)}$	$\sigma^{-2}\psi_{\omega}^{(m)}$	$\sigma^{-2}\psi_{\omega}^{(m)}$	$\sigma^{-2}\psi_{\omega}^{(m)}$
2	1	0.23996	0.25864	0.27367	0.28503
	2	0.81065	0.85123	0.88246	0.90548
3	1	0.09638	0.10456	0.11086	0.11557
	2	0.22212	0.23765	0.24960	0.25826
	3	0.39436	0.41765	0.43554	0.44863
4	1	0.05147	0.05610	0.05967	0.06229
	2	0.10969	0.11862	0.12538	0.13021
	3	0.17709	0.19000	0.19981	0.20689
	4	0.25806	0.27504	0.28802	0.29740
5	1	0.03187	0.02768	0.03730	0.03842
	2	0.06611	0.07197	0.07634	0.07946
	3	0.10328	0.11181	0.11821	0.12278
	4	0.14436	0.15541	0.16376	0.16971
	5	0.19089	0.20448	0.21479	0.22223

Table 7: $V_3 = Var(U_{1:n}^{(m)})$ and $V_4 = Var(U_{2:n}^{(m)})$

		$\theta = 0.50$		$\theta = 1$	
m	n	V_3	V_4	V_3	V_4
2	5	1.20825	0.22504	0.23615	0.24031
	10	0.50206	0.10333	0.09721	0.11088
	15	0.31526	0.06714	0.06083	0.07215
	20	0.22954	0.04973	0.04421	0.05349
	30	0.14858	0.03275	0.02856	0.03525
	40	0.10981	0.02442	0.02109	0.02629
	60	0.07214	0.01618	0.01384	0.01743
	80	0.05371	0.01210	0.01030	0.01304
	100	0.04278	0.00967	0.00820	0.01041
3	5	0.91844	0.20162	0.17160	0.21572
	10	0.36945	0.09276	0.06796	0.09996
	15	0.23016	0.06038	0.04212	0.06522
	20	0.16697	0.04478	0.03048	0.04842
	30	0.10771	0.02953	0.01961	0.03196
	40	0.07948	0.02203	0.01445	0.02386
	60	0.05213	0.01461	0.00950	0.01580
	80	0.03878	0.01093	0.00704	0.01184
	100	0.03088	0.00873	0.00560	0.00946
4	5	0.80991	0.19328	0.14754	0.20701
	10	0.30758	0.08809	0.05455	0.09523
	15	0.18913	0.05730	0.03329	0.06213
	20	0.13640	0.04249	0.02393	0.04613
	30	0.08751	0.02802	0.01530	0.03046
	40	0.06441	0.02090	0.01124	0.02274
	60	0.04214	0.01386	0.00734	0.01509
	80	0.03131	0.01037	0.00545	0.01129
	100	0.02491	0.00828	0.00433	0.00902
5	5	0.77759	0.19089	0.14033	0.20448
	10	0.27147	0.08546	0.04681	0.09190
	15	0.16431	0.05550	0.02800	0.05785
	20	0.11771	0.04114	0.01994	0.04164
	30	0.07507	0.02712	0.01264	0.02639
	40	0.05509	0.02023	0.00925	0.01922
	60	0.03595	0.01341	0.00602	0.01241
	80	0.02668	0.01004	0.00446	0.00915
	100	0.02121	0.00802	0.00354	0.00724

Table 7: Continued

		$\theta = 1.50$		$\theta = 2$	
m	n	V_3	V_4	V_3	V_4
2	5	0.08964	0.25245	0.04524	0.26157
	10	0.03671	0.11692	0.01848	0.12147
	15	0.02293	0.07617	0.01153	0.07920
	20	0.01665	0.05650	0.00837	0.05877
	30	0.01075	0.03726	0.00540	0.03878
	40	0.00793	0.02780	0.00398	0.02893
	60	0.00520	0.01843	0.00261	0.01919
	80	0.00387	0.01379	0.00194	0.01436
	100	0.00308	0.01101	0.00155	0.01147
3	5	0.06332	0.22657	0.03135	0.23449
	10	0.02487	0.10551	0.01227	0.10961
	15	0.01537	0.06895	0.00758	0.07171
	20	0.01111	0.05122	0.00547	0.05331
	30	0.00714	0.03384	0.00352	0.03524
	40	0.00526	0.02527	0.00259	0.02632
	60	0.00344	0.01677	0.00169	0.01750
	80	0.00256	0.01255	0.00126	0.01308
	100	0.00204	0.01003	0.00100	0.01045
4	5	0.05364	0.21745	0.02633	0.22499
	10	0.01954	0.10067	0.00954	0.10459
	15	0.01187	0.06581	0.00580	0.06848
	20	0.00851	0.04892	0.00417	0.05094
	30	0.00543	0.03233	0.00266	0.03370
	40	0.00398	0.02415	0.00196	0.02518
	60	0.00260	0.01604	0.00128	0.01673
	80	0.00193	0.01200	0.00095	0.01252
	100	0.00153	0.00959	0.00075	0.01001
5	5	0.05077	0.21479	0.02486	0.22223
	10	0.01656	0.09800	0.00799	0.10178
	15	0.00989	0.06406	0.00470	0.06648
	20	0.00705	0.04764	0.00331	0.04938
	30	0.00448	0.03151	0.00207	0.03261
	40	0.00329	0.02355	0.00150	0.02435
	60	0.00214	0.01564	0.00097	0.01616
	80	0.00159	0.01171	0.00071	0.01209
	100	0.00126	0.00936	0.00057	0.00966

4. Conclusions

The peculiarity of this estimation method is that, if we have the best linear unbiased estimator based on observation of size m as the kernel, then the evaluation of moments of order statistics of sample sizes up to $2m-1$ coming from standard form of (1.2) alone necessary to obtain the variances of the U-statistics defined in (3.5) and (3.6).
For example, in the case of $m=5$, by using the best linear unbiased estimators of μ and σ given in (2.1) and (2.2) respectively , one only needs the moments of order statistics arising from the standard of (1.2) for sample sizes up to 9 to obtain the U-statistic estimators for μ and σ and its variances for any sample of size n and for any given value of θ. Using the values of variances and co-variances of order statistics (given in Table 2) and the coefficients of, BLUEs of μ and σ (given in Table 3 and Table 4), we have obtained the values of $\xi_{\omega}^{(m)}$and $\psi_{\omega}^{(m)}$for $\omega=1,2,\cdots,m-1$, $m=2(1)5$ and $\theta=(0.50)(0.50)2$ (given in Table 5 and Table 6). Also, we have evaluated the variances of the U-statistic estimators for μ and σ which are given in Table 7. For practicing statisticians the results derived in the paper will be helpful, when they look for estimators of parameters of Lindley distribution using ordered random variables.

Acknowledgements

The authors express their gratefulness for the constructive criticism of the learned referees which helped to improve considerably the revised version of the paper.

REFERENCES

ALI, S., ASLAM, M., KAZMI, S. M., (2013). A study of the effect of the loss function on Bayes estimate, posterior risk and hazard function for Lindley distribution. Appl Math Model, 37, pp. 6068–6078.

ELBATAL, I., ELGARHY, M., (2013). Transmuted quasi Lindley distribution: a generalization of the quasi Lindley distribution. Int J Pure Appl Sci Technol, 18, pp. 59–70.

GHITANY, M, E., ATIEH, B., Nadarajah, S., (2008). Lindley distribution and its applications. Mathematics and Computers in Simulation, 78 (4), pp. 493–506.

GHITANY, M, E., Al-MUTAIRI, D, K., BALAKRISHNAN, N., Al-ENEZI, L, J., (2013): Power Lindley distribution and associated inference.Computational Statistics and Data Analysis, 64, pp. 20–33.

GÓMEZ, D, E., OJEDA, E, C., (2011). The discrete Lindley distribution: Properties and applications, Journal of Statistical Computation and Simulation, 81 (11), pp. 1405–1416.

HOEFFDING, W., (1948). A class of statistics with asymptotically normal distributions, The Annals of Mathematical Statistics, 19, pp. 293–325.

IRSHAD, M, R., MAYA, R., (2017). Extended Version of Generalized Lindley Distribution. South African Statistical Journal, 51, pp. 19–44.

KADILAR, G, O., CAKMAKYAPAN, S., (2016). The Lindley family of distributions: Properties and applications, Hacettepe University Bulletin of Natural Sciences and Engineering Series B: Mathematics and Statistics, 46(116), pp. 1–28.

LINDLEY, D, V., (1958). Fiducial distributions and Bayes' theorem, Journal of the Royal Statistical Society, Series B, 20 (1), pp. 102–107.

LOYD, E, H., (1952). Least-squares estimation of location and scale parameters using order statistics. Biometrical Journal, 39, pp. 88–95.

NADARAJAH, S., BAKOUSH, H., TAHMASBI, R., (2011). A generalized lindley distribution, Sankhya B - Applied and Interdisciplinary Statistics, 73, pp. 331–359.

NEDJAR, S., ZEHDOUDI, H., (2016). On gamma Lindley distribution: Properties and simulations, Journal of Computational and Applied Mathematics, 298, pp. 167–174.

SANKARAN, M., (1970). The discrete Poisson-Lindley distribution, Biometrics, 26, pp. 145–149.

SHIBU, D, S., IRSHAD, M, R., (2016). Extended New Generalized Lindley Distribution. Statistica, 76, pp. 42–56.

SULTAN, K, S., AI-THUBYANI, W, S., (2016). Higher order moments of order statistics from the Lindley distribution and associated inference, Journal of Statistical computation and Simulation, 86, pp. 3432–3445.

THOMAS, P, Y., SREEKUMAR, N, V., (2004). Estimation of the scale parameter of generalized exponential distribution using order statistics, Calcutta Statistical Association Bulletin, 55, pp. 199–208.

THOMAS, P, Y., SREEKUMAR, N, V., (2008). Estimation of location and scale parameters of a distribution by U-statistics based on best linear functions of order statistics, Journal of Statistical Planning and Inference, 138, pp. 2190–2200.

ZAKERZADEH, H., DOLATI, A., (2009). Generalized Lindley distribution, Journal of Mathematical Extension, 3 (2), pp. 13–25.

APPENDIX

*Here, we use MATHCAD software for all numerical computations. Tables 1 and 2 summarize the values of means, variances and covariances of the order statistics arising from the standard form of (1.2) for $n = 2(1)10$ and for different values of θ. For calculating the U-statistic estimators and its variances, we want the coefficients of **X** in the best linear unbiased estimators of μ and σ and the corresponding variances. Using the moments of order statistics given in Tables 1 and 2 and using the formulas defined in (2.1) and (2.3), we have evaluated the coefficients of **X** in the BLUE of μ and its variances for different values of θ and it is given in Table 3. The moments of order statistics given in Tables 1 and 2 and using the formulas defined in (2.2) and (2.4), we have evaluated the coefficients of **X** in the BLUE of σ and its variances for different values of θ which are given in Table 4.*

*For computing the numerical values of variances of the U-statistic estimators defined in (3.5) and (3.6), first we want the numerical values of $\xi_\omega^{(m)}$ and $\psi_\omega^{(m)}$ for different values of m and ω. For example if we want to calculate the numerical values of $\xi_\omega^{(m)}$ we use the formula $w_k = \binom{m+k}{m}(b'_{m+k}V_{m+k}b_{m+k})\sigma^2 - \xi_m^{(m)}, k = 1, 2, \cdots, m-1$. In particular, if for the case of $m = 2$, $\omega = 2$ and $\theta = 0.50$, we want to calculate only two values $\xi_1^{(2)}$ and $\xi_2^{(2)}$, where $\xi_2^{(2)}$ is nothing but the value of the variance. From Table 2, it is obtained as 5.77647. Using the formula w_k, the value of $\xi_1^{(2)}$ reduces to $\xi_1^{(2)} = \frac{3}{2}U - \frac{1}{2}\xi_2^{(2)}$, where $U = bVb'$, $b = \left(\frac{2a_1}{3}, \frac{a_1}{3} + \frac{a_2}{3}, \frac{2a_2}{3}\right)$, a_1 and a_2 are coefficients of **X** in BLUE of μ of order statistics of sample of size 2 and V is the variance covariance matrix of order 3. From Table 2, the matrix V is obtained as*

$$V = \begin{bmatrix} 1.26895 & 1.10362 & 0.97997 \\ 1.10362 & 3.03236 & 2.71450 \\ 0.97997 & 2.71450 & 8.76918 \end{bmatrix}.$$

*Also from Table 3, the value of a_1 is 1.65385 and that of a_2 is -0.65385. Using these values, we can easily obtain the value of $\xi_1^{(2)}$. In the same way, we can easily obtain the values of $\xi_3^{(\omega)}$, $\xi_4^{(\omega)}$ and $\xi_5^{(\omega)}$ for various values of ω and we have evaluated all these values which are given in Table 5. The Table 6 comprises the values of $\psi_\omega^{(m)}$ for different combinations of m and ω. The values of $\psi_\omega^{(m)}$ are obtained when we follow the same steps for obtaining the values of $\xi_\omega^{(m)}$, the only change is that instead of using the coefficients of **X** in the best linear unbiased estimator of μ and its variance, here we use the coefficients of **X** in the best linear unbiased estimator of σ and its variance. The numerical values of the variances of U-statistic estimators defined in, (3.5) and (3.6) are given in Table 7 for various values of parameters. If we put $m = 2$ the formula (3.5) reduces the following way for various values of n. That is*

$Var[U_{1:5}^{(2)}] = \frac{6\xi_1^{(2)}+\xi_2^{(2)}}{10}$, $Var[U_{1:10}^{(2)}] = \frac{16\xi_1^{(2)}+\xi_2^{(2)}}{45}$, $Var[U_{1:15}^{(2)}] = \frac{26\xi_1^{(2)}+\xi_2^{(2)}}{105}$,
$Var[U_{1:20}^{(2)}] = \frac{36\xi_1^{(2)}+\xi_2^{(2)}}{190}$, $Var[U_{1:30}^{(2)}] = \frac{56\xi_1^{(2)}+\xi_2^{(2)}}{435}$, $Var[U_{1:40}^{(2)}] = \frac{76\xi_1^{(2)}+\xi_2^{(2)}}{780}$,

$Var[U_{1:60}^{(2)}] = \frac{116\xi_1^{(2)}+\xi_2^{(2)}}{1770}$, $Var[U_{1:80}^{(2)}] = \frac{156\xi_1^{(2)}+\xi_2^{(2)}}{3160}$ *and* $Var[U_{1:100}^{(2)}] = \frac{196\xi_1^{(2)}+\xi_2^{(2)}}{4950}$.

Similarly we can find the values of $Var[U_{1:n}^{(3)}]$, $Var[U_{1:n}^{(4)}]$ *and* $Var[U_{1:n}^{(5)}]$ *for various values of* n. *Proceeding in the similar manner we can easily find the values of* $Var[U_{2:n}^{(m)}]$ *for different values of* m *and* n.